This exam consists of **three problems**, worth a total of 40 points.

Problem 1 has one part, and is worth 15 points.
Problem 2 has three parts, and is worth 15 points.
Problem 3 has one part, and is worth 10 points.

Below are a few preliminaries and definitions that you may find useful for the exam.

- For an N-point discrete-time sequence $x[n]$ with $x[n] = 0$ for $n < 0$ and $n > N-1$, we denote its N-point discrete Fourier Transform (DFT) by $X[k]$, where $X[k] = 0$ for $k < 0$ and $k > N-1$. The analysis and synthesis equations are:

$$\begin{aligned} \text{Analysis equation:} \quad X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad 0 \le k \le N-1, \\ \text{Synthesis equation:} \quad x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad 0 \le n \le N-1, \end{aligned}$$

where $W_N \triangleq e^{-j(2\pi/N)}$.

- For a general discrete time sequence $x[n]$, we denote its discrete-time Fourier Transform (DTFT), when it exists, by $X(e^{j\omega})$. The analysis and synthesis equations are:

$$\begin{aligned} \text{Analysis equation:} \quad X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ \text{Synthesis equation:} \quad x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega. \end{aligned}$$

Similarly, the z-transform of a discrete-time signal $x[n]$ is given by $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$.

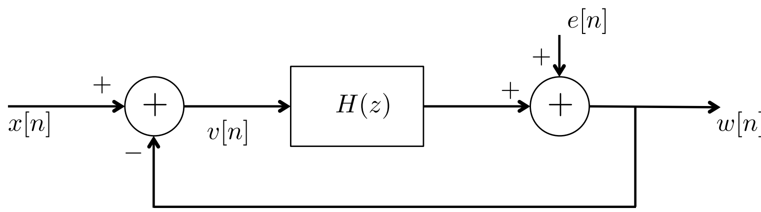
1) **[15 points]** Let $X(e^{j\omega})$ be the discrete time Fourier Transform (DTFT) of the discrete-time signal $x[n] = (1/2)^n u[n]$. Find a length-5 sequence $g[n]$ whose five-point discrete Fourier transform (DFT) $G[k]$ is identical to the samples of the DTFT of $x[n]$ at $\omega_k = 2\pi k/5$, i.e.,

$$g[n] = 0 \quad \text{for} \quad n < 0,\ n > 4$$

and

$$G[k] = X(e^{j2\pi k/5}) \quad \text{for} \quad k = 0, 1, \ldots, 4.$$

2) **[15 points total, 3 parts]** In the figure below, $H(z)$ is the system function of a causal linear time invariant (LTI) system.



a) **[5 points]** Using z-transforms of the signals shown in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.

b) **[5 points]** For the special case $H(z) = z^{-1}/(1 - z^{-1})$, determine $H_1(z)$ and $H_2(z)$.

c) **[5 points]** Is the system $H(z)$ stable? Are the systems $H_1(z)$ and $H_2(z)$ stable?

3) **[10 points]** Suppose $x[n]$ is a finite-duration discrete-time signal that is known to be zero for all $n < 0$.

Show that if $x[n]$ is also *binary-valued* (i.e., its nonzero components can only take the value 1) it can be recovered exactly from its z-transform evaluated at the *single point* $z = 1/2$. That is, provide a simple strategy for recovering causal, binary-valued signals $x[n]$ from $X(z)|_{z=1/2}$.